Mathematisches Forschungsinstitut Oberwolfach

Report No. 58/2017

DOI: 10.4171/OWR/2017/58

Mathematical Instruments between Material Artifacts and Ideal Machines: Their Scientific and Social Role before 1950

Artifacts for geometrical transformations and drawing curves in the classrooms, workshops and exhibitions

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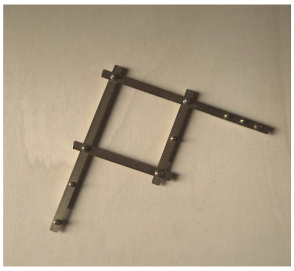
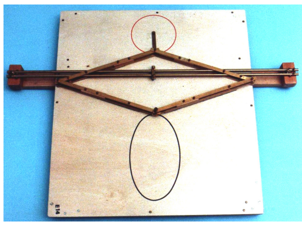
This contribution focuses on the use of some instruments, called "mathematical machines", in teaching and learning mathematics. In particular, we refer to some machines which have been built with a didactical aim by secondary school teachers on the basis of their descriptions in historical texts, ranging from classical Greek mathematics (linked to the theory of conic sections) to 20th century mathematics. These machines are currently collected in the rooms of the Laboratory of Mathematical Machines (MM-Lab, www.mmlab.unimore.it) at the University of Modena and Reggio Emilia. The MMLab works for both mathematics education research and popularization of mathematics ([1]).

A mathematical machine (related to geometry) is an artifact designed and built for the following purpose: it aims at forcing a point, a line segment or a plane figure to move or to be transformed according to a mathematical law that has been determined by the designer. A well-known mathematical machine is a pair of compasses. In the educational approach to the use of mathematical machines, the interest concerns not only what an instrument allows to make, but also how students using a machine can construct mathematical meanings embedded in the machine itself (and related to its structure and functioning) and justify its function. This paper mainly concerns pantographs for geometrical transformations and drawers for conic sections.

The history of instruments for geometry other than the compass and straightedge started with Descartes' work. In his *Géométrie* ([3]) Descartes studied curves that were mechanically obtained and worked to obtain their algebraic expression. This is the case of the hyperbola drawer. The instruments were above all considered as theoretical ones: Descartes thought about the curve as drawn by imaginary movements. However, Descartes described other instruments from a different perspective in other books. For instance, in the *Dioptrique* ([3]) the descriptions of the gardener's ellipse and the hyperbola drawer with tightened treads contain some practical elements for the user and their drawings show a hand where the pencil had to be placed to move the thread and draw the curve. In the same book, the author also described a machine based on the theory of conic sections by Apollonius for obtaining hyperbolic shapes for smoothing lenses. All these instruments have been constructed and used with secondary school and university students in several activities carried out by the MMLab.

In the development of the geometry of instruments ("geometria organica"), the interest passed from the conception of specific drawers to a more theoretical approach on drawers containing linkages. Our educational interest concerns linkages with one degree of freedom, corresponding to curve drawers, and linkages with two degrees of freedom, corresponding to pantographs. The latter ones are local instruments, in the sense that they determine a correspondence between limited plane regions, while geometric transformations are defined, globally, for all the points of the plane. An interesting example of pantograph was proposed by C. Scheiner in his *Pantographice, seu ars delineandi* ([8]). This author constructed an instrument called "linear parallelogram"(see figure on the left). This articulated figure is fixed in a plane by a point and can make homothetic transformations. As Scheiner showed on the title page of his book (see figure in the center), this pantograph works on the plane but it is also a component of a perspectograph for perspective drawings with the characteristic of drawing enlarged perspective images. Scheiner's idea of using some figures, in particular quadrilaterals, as components of other instruments was very fruitful. For instance, if two opposite vertices of an articulated rhombus are put into a groove and two pencils are inserted into the two free vertices, a mathematical machine for reflection is obtained. In our educational perspective, variations in the structure of a machine are important for fostering conjectures and argumentations by the students. For instance, if two points are chosen at the same distance from a free vertex of the rhombus and they are put into the groove, does the new machine always make a reflection? The answer is quite intriguing: the two free vertices are corresponding points in an affine transformation ([7]). This kind of instrument (see figure on the right) was proposed by M.N. Delaunay for drawing an ellipse by a transformation of a circle ([2]).





Scheiner's contribution was also commented by G. Koenigs in his *Leçons de cinématique*: "La théorie des systèmes articulés ne date que de 1864. Sans doute on les a utilisés bien avant cette époque; il se peut même que quelque esprit amoureux de précision rétrospective découvre des systèmes articulés dans l'antiquité la plus reculée; nous apprendrions une fois de plus que tout siècle détient inconsciemment entre ses mains les découvertes des siècles futurs, et que l'histoire des choses devance très souvent celle des idées. Lorsque, en 1631, le P. Scheiner publia pour la première fois la description de son pantographe, il ne connut certainement pas l'idée générale dont son petit appareil n'était qu'une manifestation naissante; on peut même affirmer qu'il ne pouvait pas la connaître, car cette idée tient la notion élevée de la transformation des figures, notions qui appartient à notre siècle et donne un caractère uniforme à tous les progrès qu'il a vus s'accomplir. Le mérite de PEAUCELLIER, de KEMPE, de HART, de LIPKINE est moins d'être parvenu à tracer avec des systèmes articulés telle ou telle courbe particulière, que d'avoir aperçu les moyens de réaliser avec ces systèmes de véritables transformations géométriques. Dans cette remarque réside ce qu'il y a de vraiment général dans la théorie des systèmes articulés" ([4, p. 243]).

Pantographs for geometrical transformations are exploited in Italian educational projects in school from 7-grade students to 10-grade students. Conic sections drawers are proposed in teaching experiments concerning a synthetic approach to conic sections in secondary school (grade 11). All these projects are based on the methodology of mathematics laboratory ([5]), in which students work in a small group with the mathematical machines, participate to mathematical discussions, but also solve some individual tasks. The teacher acts as a cultural mediator, who constructs tasks involving the mathematical machines related to a chosen mathematical content (he/she used a machine as tool of semiotic mediation) and manages collective discussions. From the perspective of mathematics education, the aim of the design research on conic section is to study if and how the mathematical machines can be used for defining the conic sections and for looking to their properties, from a unifying vision of the curves. In that sense, the mathematical machines are involved by two didactical functionalities: introducing and defining a particular conic section and fostering argumentation and proving processes. The educational path consists of four parts (20 hours): 1) an introduction to linkages by the exploration of Van Schooten's compass; 2) the exploration of conic drawers with tightened threads (ellipse, parabola and hyperbola) for looking for the definition of these curves; 3) the exploration of conic drawers with crossed parallelograms (ellipse and hyperbola) for fostering argumentation and proving processes; 4) a historical brief survey on conic sections: the definitions of conic sections by Menaechmus and Apollonius; the description of a perfect compass and Descartes's machine for hyperbolic lenses; Dandelin's Theorem and its proof.

This contribution ends with the presentation of the activities of the MMLab ([6]). In the MMLab equipped rooms, we propose laboratory sessions to secondary school classes and to groups of university students. The topics are: conic sections and conic drawers, geometrical transformations, perspective, and the problem of the angle trisection. Each session takes approximately two hours and covers three steps (historical introduction, group work on mathematical machines, collective presentation of each group work). The MMLab also participates in cultural events in Modena, above all with the permanent exhibition on perspective, and to other exhibitions in other towns, in Italy and abroad.

References

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